ON THE CONVERGENCE OF THE SERIES USED IN THE SUBJECT OF PERTURBATIONS.

BY. DR. G. W. HILL.

THE perturbations of the planets and the coördinates of the moon have been developed by astronomers in infinite series of terms involving sines or cosines of linear functions of two or more arguments with positive or negative integral multipliers. These arguments vary proportionally with the time, and their periods, in accordance with notions derived from the theory of probabilities, are supposed to be incommensurable with each other. Recently M. Poincaré has much insisted that, under the latter condition, these series, in the rigorous mathematical sense, are divergent (*Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. II., pp. 277–280). However, the reasons brought forward to sustain this opinion are scarcely convincing, and I think there has been some scepticism among astronomers in reference to the matter. Without attempting to find any flaw in M. Poincaré's logic, I simply wish to point out a class of cases where the convergency of the series can be shown in spite of the incommensurability of the component arguments.

In many problems of dynamics, where the integral of conservation of areas has place, we shall often have the longitude λ of the moving point given by a quadrature. We choose as our example the equation

$$\frac{1}{\mathrm{n}}\frac{d\lambda}{dt} = \sum_{i=0}^{i=\infty}\sum_{i'=-\infty}^{i'=+\infty} a^{i+|i'|}\cos(il+i'l'), \qquad (1)$$

in which $l = nt + c$ and $l' = n't + c'$, and a is a positive constant less than unity. Here λ corresponds to M. Poincaré's $\log x$ (p. 279 of the above-quoted volume). Under the condition named, the series of (1) is convergent. Now let both members of the equation be integrated; putting μ for $\frac{n'}{n}$, we have

$$\lambda = \epsilon + \mathrm{n}t + \frac{\mathrm{n}}{n}\Sigma\frac{1}{i+i'\mu}a^{i+|i'|}\sin(il+i'l'), \qquad (2)$$

ϵ being the added arbitrary constant, and the sign of summation Σ having the same extension as that of the double sign in (1), except that the combination $i = i' = 0$ is omitted. When

μ is an irrational quantity, the summation of this equation constitutes a divergent series according to M. Poincaré.

We prefer to write (2) in the more expanded form

$$\lambda = \epsilon + \mathrm{n}t + \frac{\mathrm{n}}{n}\left\{\Sigma\frac{1}{i}\alpha^i\left[\sin il + \frac{1}{\mu}\sin il'\right]\right.$$
$$+ 2\Sigma\Sigma\frac{i}{i^2-i'^2\mu^2}\alpha^{i+i'}\sin il\cos i'l'$$
$$\left.- 2\mu\Sigma\Sigma\frac{i'}{i^2-i'^2\mu^2}\alpha^{i+i'}\cos il\sin i'l'\right\}, \qquad (3)$$

where the extent of the summation, in all cases, is from 1 to $+\infty$. This will be the signification of the sign Σ hereafter.

Before we proceed to consider the question of convergence in reference to (3), it may be of interest to point out that the series of (1) admits of summation. For, by arranging it according to cosines of multiples of l', we have

$$\begin{aligned}\frac{1}{\mathrm{n}}\frac{d\lambda}{dt} &= [1 + \alpha\cos l + \alpha^2\cos 2l + \alpha^3\cos 3l + \cdots]\\ &+ [1 + 2\alpha\cos l + 2\alpha^2\cos 2l + 2\alpha^3\cos 3l + \cdots]\alpha\cos l'\\ &+ [1 + 2\alpha\cos l + 2\alpha^2\cos 2l + 2\alpha^3\cos 3l + \cdots]\alpha^2\cos 2l'\\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\ &= [1 + \alpha\cos l + \alpha^2\cos 2l + \cdots]\\ &+ [1 + 2\alpha\cos l + 2\alpha^2\cos 2l + \cdots][\alpha\cos l' + \alpha^2\cos 2l' + \cdots]\\ &= \frac{1 + \alpha^4 - \alpha(1+\alpha^2)(\cos l + \cos l') + 2\alpha^2\cos l\cos l'}{(1 - 2\alpha\cos l + \alpha^2)(1 - 2\alpha\cos l' + \alpha^2)}.\end{aligned}$$

Then λ can be expressed by the following quadrature:

$$\lambda = \epsilon + \mathrm{n}t +$$
$$\frac{\mathrm{n}}{2}\int_0^t\left[\frac{(1-\alpha^2)^2}{[1-2\alpha\cos(nt+c)+\alpha^2][1-2\alpha\cos(n't+c')+\alpha^2]} - 1\right]dt. \qquad (4)$$

Our supply of functions in the integral calculus is inadequate to the expression of this quadrature in finite terms; but there is no bar to our finding the amount of motion of λ between any two given times t_0 and t_1 by the process of mechanical quadratures.

Quadratures may also be invoked to aid in the expression of (3). For, by putting

$$X_i = \frac{1}{1-i^2\mu^2}\alpha\cos l + \frac{1}{4-i^2\mu^2}\alpha^2\cos 2l + \frac{1}{9-i^2\mu^2}\alpha^3\cos 3l + \cdots, \qquad (5)$$

as well as $\qquad P_i = -\frac{dX_i}{dl}, \quad Q_i = iX_i,$

(3) takes the form

$$\lambda = \epsilon + nt + \frac{n}{n}\left\{\Sigma\frac{1}{i}\alpha^i\left[\sin il + \frac{1}{\mu}\sin il'\right]\right.$$
$$+ 2(P_1\alpha\cos l' + P_2\alpha^2\cos 2l' + P_3\alpha^3\cos 3l' + \cdots)$$
$$\left. - 2\mu(Q_1\alpha\sin l' + Q_2\alpha^2\sin 2l' + Q_3\alpha^3\sin 3l' + \cdots)\right\}. \quad (6)$$

But it will be perceived that, putting

$$L = \frac{\alpha\cos l - \alpha^2}{1 - 2\alpha\cos l + \alpha^2},$$

X_i satisfies the linear differential equation of the second order

$$\frac{d^2X_i}{dl^2} + i^2\mu^2X_i + L = 0. \quad (7)$$

By the integration of this, we have

$$X_i = \cos(i\mu l)\int L\sin(i\mu l)dl - \sin(i\mu l)\int L\cos(i\mu l)dl. \quad (8)$$

The first summation of (6) may be obtained through the use of the well-known equation

$$\Sigma\frac{1}{i}\alpha^i\sin il = -\tfrac{1}{2}l + \arctan\left[\frac{1+\alpha}{1-\alpha}\tan\frac{l}{2}\right]. \quad (9)$$

We come now to the consideration of the question of convergence of the two double summations in (3). In these we may put

$$\sin il = \cos il = \sin i'l' = \cos i'l' = 1;$$

the matter at issue is not thereby changed. Hence it suffices to determine the convergence or divergence of the two series

$$\Sigma\Sigma\frac{i}{i^2 - i'^2\mu^2}\alpha^{i+i'}, \quad \Sigma\Sigma\frac{i'}{i^2 - i'^2\mu^2}\alpha^{i+i'}.$$

It is necessary now to specify the precise nature of the quantity μ. As an example, we assume that $\mu = \sqrt{h}$, h being a non-square integer. The divisor in the two series is then $i^2 - hi'^2$. From the theory of indeterminate equations of the second degree, we learn that the least absolute value of this expression is unity; that is, we may write

$$|i^2 - hi'^2| \geqq 1.$$

Unity may therefore be substituted for this divisor in the summations just given without thereby modifying the question of their convergence, which is thus narrowed to the convergence or divergence of the single expression

$$\Sigma\Sigma i\alpha^{i+i'}.$$

But this series is convergent, being equivalent to the product

$$[\Sigma i\alpha^i][\Sigma\alpha^i]=[\alpha+2\alpha^2+3\alpha^3+\cdots][\alpha+\alpha^2+\alpha^3+\cdots].$$

If we agree to take j terms of each factor of this as an approximation, the error committed will be

$$\frac{j(1-\alpha)+2}{(1-\alpha)^3}\alpha^{j+2},$$

which may be made as small as we please by taking j sufficiently large. This expression is also a superior limit to the error committed in either of the double summations of (3) when the series is pushed to the same extent in reference to the varying integers i and i'.

As a more general example, including the former, we will take

$$\mu^2=\frac{p}{q}+\sqrt{\frac{p'}{q'}},$$

where p, q, p', and q' are integers, and $\frac{p'}{q'}$ is not an exact square. If we substitute this value of μ^2 in the expression $i^2-i'^2\mu^2$, and rationalize and render integral this denominator, multiplying it by the proper factor, we find that it becomes

$$q'(q^2i^2-p^2i'^2)^2-p'q^4i'^4.$$

Now this expression, which is integral, cannot vanish, for this would make $\sqrt{\frac{p'}{q'}}$ rational; consequently, its absolute value is at least unity. We may then substitute unity for it in the summations we consider without affecting the question of their convergence. Thus, the latter is seen to depend on the convergence of

$$\Sigma\Sigma i^s\alpha^{i+i'} \text{ and } \Sigma\Sigma i^2i'^t\alpha^{i+i'}.$$

As these summations are quite plainly convergent, there is nothing further to be said.

As a still greater generalization, let us suppose that μ is an irrational root of an algebraic equation with rational coefficients. Then, in a similar way as before, $i^2-i'^2\mu^2$ may be rationalized and rendered integral by multiplying by the proper factor. The absolute value of the thus modified denominator is at least unity. On consulting the form of the numerator, it is gathered that the convergence of our series depends on that of various summations whose general type is

$$\Sigma\Sigma i^{\nu}i'^{\nu'}\alpha^{i+i'},$$

where ν and ν' are finite positive integers. The convergence is therefore established.

When μ is rational, we would simply transfer the terms which, in the integration, become proportional to t to the term nt of (3). Then the remainder would constitute a periodic and convergent series. Thus, in all cases where μ is a root of an algebraic equation with rational coefficients, (3) may be affirmed to be a convergent series.

Our method of treatment cannot be applied in the case where μ is a root of a transcendental equation. But it may be remarked that the higher the degree of the equation which has μ for a root, the larger become the exponents ν and ν'. Thus, one is led to think that, when μ is a root of a transcendental equation, these exponents become infinite. Should this be correct, the summations, whose general type has just been given, become divergent. But we would not be warranted in concluding thence the divergence of (3). The whole question turns on the properties of the integral

$$\int\left[\frac{(1-\alpha^2)^2}{[1-2\alpha\cos(nt+c)+\alpha^2][1-2\alpha\cos(n't+c')+\alpha^2]}-1\right]dt.$$

It is possible that the ratio $\frac{n'}{n}$ may have values which would make this expression tend towards infinity as the limits of integration were removed farther from each other. But I am not aware that this has been proved. But it is something gained to have established that, when $\frac{n'}{n}$ is an irrational root of an algebraic equation with rational coefficients, the expression is always contained between finite limits, whatever may be the limits of integration.

Our conclusions still hold when in (1) we substitute the general coefficient $C_{i,i'}$ for $\alpha^{i+|i'|}$, provided we have the condition

$$|C_{i,i'}| \lessgtr \alpha^{i+|i'|},$$

α being positive and less than unity. Also, we might assume a different rate of decrement in the coefficients with augmenting multiples of l' from that which belongs to l. Calling this α', for $\alpha^{i+|i'|}$ we should have $\alpha^i\alpha'^{|i'|}$, and the course of reasoning would be scarcely changed by this modification. In case there are more than two elementary arguments, the mode of proceeding is quite similar. The ratios $\frac{n'}{n}$, $\frac{n''}{n}$, $\frac{n'''}{n}$, etc., being irrational roots of algebraical equations, the divisors introduced by integration must be rationalized and rendered integral by multiplying both terms of the fraction by the proper factor. The convergence of the series is made out as before.

ON THE LOCUS OF THE FOCI OF CONICS HAVING DOUBLE CONTACT WITH TWO FIXED CONICS.

BY R. A. ROBERTS.

THE system of conics having double contact with two conics includes as special cases the two systems of conics passing through four fixed points and touching four fixed lines, respectively. These latter result from making the fixed conics break up into factors, in the first case, in the tangential sense, and in the second case directly. Thus the general case includes and combines together into one whole the properties of the two special cases. Now I propose to consider the locus of the foci of the general system. This must include as special cases the known loci in the case of conics through four fixed points and touching four fixed lines. The first is a curve of the sixth order, which, when the points are concyclic, as has been shown by Professor Sylvester, breaks up into the two circular cubics having the points for foci, and the second is a circular cubic having its double focus on itself and passing through all the intersections of the four lines.

Now foci, being the intersections of tangents drawn to the curve from the imaginary points at infinity, are most easily treated by tangential coördinates, which I consequently use, the equations being precisely the same as the direct ones; viz. as given by Salmon, a conic having double contact with S, S' is, if $S - kS' = EF$,

$$\theta^2 E^2 - 2\theta(S + kS') + F^2 = 0.$$

But this being now a tangential equation, we may put

$$E = \lambda x_1 + \mu y_1 + \nu,$$

$$F = \lambda x_2 + \mu y_2 + \nu,$$

$$S + kS' = A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu^2 + 2G\nu\lambda + 2H\lambda\mu,$$

where x, y are rectangular Cartesian coördinates. Now the foci are the points of intersection of the tangents drawn parallel to $x \pm iy$. Hence for the foci we may put $\nu = x \pm iy$, $\lambda = -1$, $\mu = \mp i$; we thus get

$$\theta^2 (u-a)^2 - 2C\theta(u-c)(u-d) + (u-b)^2 = 0,$$

$$\theta^2 (v-a')^2 - 2C\theta(v-c')(v-d') + (v-b')^2 = 0, \qquad (1)$$

where $u = x + iy$, $v = x - iy$, and a, b, c, d are values of u for certain points, E, F, and two others, the corresponding values

of v being a', b', c', d'. Hence, by elimination of θ, the locus is found to be, in general, a curve of the sixth degree, which may be written

$$\begin{aligned}4\{&(u-a)^2(u-b)^2-C^2(u-c)^2(u-d)^2\}\\&\times\{(v-a')^2(v-b')^2-C^2(v-c')^2(v-d')^2\}\\&=\{(u-a)^2(v-b')^2+(u-b)^2(v-a')^2\\&-2C^2(u-c)(u-d)(v-c')(v-d')\}^2,\end{aligned}$$

or
$$\begin{aligned}4C^2\{&(u-c)(u-d)(v-b')^2-(v-c')(v-d')(u-b)^2\}\\&\times\{(u-c)(u-d)(v-a')^2-(v-c')(v-d')(u-a)^2\}\\&=\{(u-a)^2(v-b')^2-(u-b)^2(v-a')^2\}^2.\end{aligned}$$

From the latter equation it is manifest that the locus is of the sixth degree, as in each of the factors the coefficient of u^2v^2 is seen to vanish; and, moreover, that the circular points at infinity, as well as the points E, F, are double points on the curve. From the same equation it is easy to see that

$$\frac{(u-a)^2}{(v-a')^2}=\frac{(u-b)^2}{(v-b')^2}=\frac{(u-c)(u-d)}{(v-c')(v-d')}$$

must also give double points, and indeed it is evident that the values of u, v which make the quadratics in θ identical must lead to such points.

Hence to find these nodes, we may write

$$(u-a)^2=k(v-a')^2,\quad (u-b)^2=k(v-b')^2,$$
$$(u-c)(u-d)=k(v-c')(v-d').$$

Thus if l, m, n are any numbers,

$$\begin{aligned}l(u-a)^2+m(u-b)^2+n(u-c)(u-d)&\\=k\{l(v-a')^2+m(v-b')^2+n(v-c')(v-d')\},&\end{aligned}$$

so that by properly selecting l, m, n, we obtain

$$Pu+Q=kC_1,$$
$$C_2=k(P'v+Q'),$$

whence
$$(Pu+Q)(P'v+Q')=C_1C_2,$$

which represents a circle. But we also have

$$(u-a)^2(v-b')^2-(u-b)^2(v-a')^2=0,$$

which breaks up into the line joining E, F, and the circle described on EF as diameter. One point on EF is irrelevant, so that three nodes are obtained in this way.

Now proceeding to the first form of the equation of the locus, we see that the lines

$$(u-a)^2(u-b)^2 - C^2(u-c)^2(u-d)^2 = 0,$$
$$(v-a')^2(v-b')^2 - C^2(v-c')^2(v-d')^2 = 0$$

are bitangents of the locus; that is, according to the definition of a focus, the intersections of the first four lines with the second four determine the foci of the curve, and, further, these points are not merely ordinary foci, but double foci of a certain kind. Now it is easily seen that

$$(u-a)(u-b) \pm C(u-c)(u-d) = 0,$$
$$(v-a')(v-b') \pm C(v-c')(v-d') = 0,$$

are precisely the equations which determine the foci of the given conics S, S'.

Hence the foci of the two fixed conics are double foci of the locus. But there are also other foci, which are of the ordinary kind, as we can see by the following considerations: the locus has seven double points; therefore by the formula

$$m = n(n-1) - 2\delta;$$

taking $n = 6$, $\delta = 7$, we see that the class of the locus is 16, so that 12 tangents can be drawn to the curve from the double point I. But the four lines to the double foci are bitangents, and count as eight ordinary tangents. Thus, four ordinary tangents can be drawn to the curve from I, and these give rise to four ordinary foci.

To find the points at infinity on the curve, we make u or v infinite in (1), and we thus get

$$\theta^2 - 2C\theta + 1 = 0.$$

This determines the two curves of the system which touch the line at infinity, or, in other words, are parabolas, as ought evidently to be the case from geometrical considerations. Hence if these two parabolas are real, the locus has two real asymptotes.

Since there are three values of k for which $S - kS'$ breaks up into factors, there are, as shown in Salmon's *Conics*, three systems of conics having double contact with S, and the determination of the locus of the foci is precisely the same in each case; that is, the entire locus of the foci consists, in general, of three sextic curves.

We now consider some cases in which the locus reduces to one of lower degree.

Solving the equations (1), we get

$$u=\frac{a\theta^2+C(c+d)\theta+b\pm\sqrt{\Theta_1}}{\theta^2-2C\theta+1},$$

$$v=\frac{a'\theta^2+C(c'+d')\theta+b'\pm\sqrt{\Theta_2}}{\theta^2-2C\theta+1}, \qquad (2)$$

where $\Theta_1=\theta\{2C(a-c)(a-d)\theta^2+2C(b-c)(b-d)$
$+[(a-b)^2-C^2(c-d)^2]\theta\},$

$\Theta_2=\theta\{2C(a'-c')(a'-d')\theta^2+2C(b'-c')(b'-d')$
$+[(a'-b')^2-C^2(c'-d')^2]\theta\}.$

From these expressions it is evident that the locus will simplify if Θ_1 and Θ_2 are proportional to each other. Now if we consider the geometrical meaning of $\Theta_1=0$, it is easy to see that the quadratic factor gives the two conics of the system which may be described to pass through the point I. Similarly, $\Theta_2=0$ gives the two conics through J. Hence if the two quadratic factors are proportional, two conics of the system pass through I, J, or, in other words, are circles. Now this condition will evidently be satisfied if the two fixed conics have a principal axis in common. Thus, there are two circles belonging to the system

$$\theta^2(x+h)^2+(x+k)^2-2\theta(ax^2+by^2+2gx+c)=0,$$

which are evidently determined by

$$\theta^2+1-2\theta(a-b)=0.$$

It is evident that the locus must be divisible by the equation of the principal axis, or rather its square, the remaining factor giving the locus of the foci lying on the other principal axis; namely, the axis major for real foci.

Taking, then, the fixed principal axis as the axis of y in the equations (2), we have $a'=a$, $b'=b$, $c'=c$, $d'=d$, and since

$$u+v=2x,\quad u-v=2iy,$$

$$x=\frac{a\theta^2+C(c+d)\theta+b}{\theta^2-2C\theta+1},$$

$$y=\frac{\sqrt{-\Theta}}{\theta^2-2C\theta+1};$$

whence, also, $\quad x^2+y^2=\dfrac{a^2\theta^2-2Ccd\theta+b^2}{\theta^2-2C\theta+1}.$

Thus eliminating θ, we get a result of the form

$$(x^2+y^2)^2+(px+q)(x^2+y^2)+rx^2+2sx+t=0,$$

which represents a bicircular quartic of which $y=0$ is an axis of symmetry. A particular case that may be considered as meriting special consideration is when the two fixed conics are confocal, and the enveloping conics are concentric with them. If a line $\lambda x+\mu y+\nu=0$ touches

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

we have $$a^2\lambda^2+b^2\mu^2-\nu^2=0,$$

and for the second conic, say, $a'^2\lambda^2+b'^2\mu^2=\nu^2$, where

$$a^2-a'^2=b^2-b'^2=h^2.$$

Then the concentric system of enveloping conics may be written

$$h^2\cos\theta(\lambda^2-\mu^2)+2h^2\sin\theta\lambda\mu=(a^2+a'^2)\lambda^2+(b^2+b'^2)\mu^2-2\nu^2.$$

Hence for the foci, putting $\lambda=1$, $\mu=i$, $\nu=-(x+iy)$, we get

$$x^2-y^2-c^2+h^2\cos\theta=0,\quad 2xy+h^2\sin\theta=0;$$

thus the locus is the Cassinian oval

$$(x^2+y^2)^2-2c^2(x^2-y^2)+c^4-h^4=0.$$

This equation, of course, also denotes that the product of the distances of a focus of the variable conic from the two fixed foci is constant.

There is also another case in which two conics of the system are circles. Consider the system

$$\theta^2P^2+Q^2-\theta(S+kS')=0;$$

then if P, Q are right angles to each other, and parallel to the axes of $S+kS'$, there are two circles of the system; for, putting $P=x$, $Q=y$, and

$$S+kS'=ax^2+by^2+2gx+2fy+c,$$

two circles are evidently given by the roots of the equation

$$\theta^2-1-\theta(a-b)=0.$$

The two fixed conics being in this case

$$ax^2+by^2\pm2xy+2gx+2fy+c=0,$$

two chords of intersection, x, y, are rectangular, and, further, the eccentricities of both are equal, as the asymptotes, real or imaginary, contain equal angles.

Now the principal axes of

$$\theta^2 x^2 + y^2 - \theta(ax^2 + by^2 + 2gx + 2fy + c) = 0$$

are evidently obtained by differentiating with regard to x and y; *i.e.* are

$$(\theta - a)x - \theta g = 0,$$
$$(1 - \theta b)y - f = 0,$$

and according as we use one or other of these equations we must obtain different loci. Hence as the complete locus is a circular curve of the sixth degree, the separate loci must be circular cubics, as could also be demonstrated from (2), when Θ_1, Θ_2 are supposed to be proportional.

Now let us consider the case in which one of the fixed conics is a circle.

One of the equations (1) which determine the foci of the locus may be written

$$\{\theta(u-a) + u - b\}^2 = 2\theta\{(u-a)(u-b) + C(u-c)(u-d)\},$$

the factors on the right side giving the foci of one of the fixed conics. Hence, if the latter is a circle, the factor must be a perfect square; that is, we may put

$$(u-a)(u-b) + C(u-c)(u-d) = 2m^2(u-e)^2.$$

Thus, if $\theta = t^2$, we have

$$t^2(u-a) + u - b \pm 2mt(u-e) = 0,$$

and similarly

$$t^2(v-a') + v - b' \pm 2mt(v-e') = 0,$$

whence

$$u = \frac{at^2 + b \pm 2met}{t^2 + 1 \pm 2mt},$$

$$v = \frac{a't^2 + b' \pm 2met}{t^2 + 1 \pm 2mt}.$$

If we take the same signs of t in u and v, we evidently get a conic; and the foci of this conic are easily seen to coincide with the foci of the fixed conic. But if we take different signs of t, viz. if we write

$$u = \frac{at^2 + b + 2met}{t^2 + 1 + 2mt},$$

$$v = \frac{a't^2 + b' - 2met}{t^2 + 1 - 2mt},$$

we obtain a unicursal bicircular quartic which has also the foci of the fixed conic as foci.

A direct method of considering the locus in this case may be worth considering. Let the tangential equation of the circle be

$$(\alpha\lambda + \beta\mu + \gamma\nu)^2 - r^2(\lambda^2 + \mu^2) = 0,$$

or $G^2 - r^2\Omega = 0$, then we may take

$$(\theta E + F)^2 - 4\theta(G^2 - r^2\Omega) = 0$$

as the tangential equation of the system of conics. Now we find the foci of a conic by combining its equation with the equation of the circular points, viz. $\Omega = 0$, so as to obtain an equation which breaks up into factors, these factors giving the foci. But putting $\Omega = 0$, we get

$$(\theta E + F)^2 - 4\theta G^2 = 0,$$

which breaks up into the factors

$$\theta E + F \pm 2G\sqrt{\theta} = 0. \qquad (3)$$

If these are real, they give the real foci, and the *envelope* with regard to θ gives the *locus* of the foci in tangential coordinates. Thus the locus is $G^2 = EF$, a conic passing through E, F, G being the intersection of the tangents at these points. Now $G^2 - r^2\Omega = EF$ is the fixed base conic, and $G^2 - EF = 0$ is confocal with this, so that the locus is a conic confocal with the base conic, as we have seen already. But further, this conic is now shown to pass through the points E, F. Thus for the three systems of variable conics the locus consists of three conics confocal with the given conic and passing, respectively, through a pair of the intersections of the common tangents of the given conic and circle.

It may be observed that Chasles originally demonstrated the theorem that the intersections of the common tangents of a conic and a circle lie by pairs on conics confocal with the given conic.

This method only gives the locus of the foci when (3) represents real foci. If these factors give imaginary foci, then the locus of the real foci consists of nodal bicircular quartics, as has been shown already.

If the two conics are circles, then by the preceding results we can easily deduce the locus of the foci. We thus find that if a conic has double contact with two fixed circles, S, S', the locus of the foci consists of a pair of circles concentric with S, another pair concentric with S', a circle coaxial with S, S', and the line joining the centres of S, S'.

These results may be easily proved directly.

Let $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

be the conic referred to its axes, and let

$$(x-\alpha)^2+y^2=r^2,$$
$$x^2+(y-\beta)^2=\rho^2,$$

be circles having double contact with the conic. Then it can be easily shown that

$$\frac{\alpha^2}{c^2}+\frac{r^2}{b^2}=1, \quad \frac{\rho^2}{a^2}-\frac{\beta^2}{c^2}=1,$$

where $c^2=a^2-b^2$.

Now if we take two circles of the first system, the foci simply lie on the line joining their centres; but if the circles are both of the second system, we have $\rho^2=\frac{a^2}{c^2}(\beta^2+c^2)=\frac{a^2}{c^2}R^2$, where R is the distance of a focus from the centre of one of the circles; also, $\rho'^2=\frac{a^2}{c^2}R'^2$ for another circle. Hence $\frac{R^2}{\rho^2}=\frac{R'^2}{\rho'^2}$, which may be written

$$\frac{R^2-\rho^2}{\rho^2}-\frac{(R'^2-\rho'^2)}{\rho'^2}=0,$$

showing that the locus is a circle coaxial with the two given circles.

If the two circles are of different systems from

$$\frac{\alpha^2}{c^2}+\frac{r^2}{b^2}=1, \quad \frac{\rho^2}{a^2}-\frac{\beta^2}{c^2}=1,$$

we get $$\frac{\alpha^2+\beta^2}{c^2}=\frac{d^2}{c^2}=\frac{\rho^2}{a^2}-\frac{r^2}{b^2}$$

if d is the distance between the centres of the circles. Hence, e being the eccentricity, we have

$$d^2=e^2\rho^2-\frac{e^2}{1-e^2}r^2,$$

so that e^2 is determined by a quadratic equation. Now we have $R^2=e^2\rho^2$, which, for the two values of e^2, gives two circles concentric with that one of the fixed circles which has its chord of contact parallel to the focal axis. Thus, for the two fixed circles, two pairs of circles constitute the locus.

Another case in which a reduction of degree occurs is when the two fixed conics are concentric, and the enveloping system

has also the same centre. If we take the points E, F at infinity in the tangential equation

$$\theta^2 E^2 + F^2 - 2\theta(A\lambda^2 + B\mu^2 + 2H\lambda\mu + C\nu^2) = 0,$$

where the origin is at the fixed centre, we may write

$$E = l\lambda + m\mu, \quad F = l'\lambda + m'\mu,$$

so that E, F are the points at infinity on the sides of a parallelogram circumscribed about both the fixed conics. The foci are then determined by

$$\theta^2(l + im)^2 + (l' + im')^2 - 2\theta(A - B + 2iH + Cu^2) = 0,$$
$$\theta^2(l - im)^2 + (l' - im')^2 - 2\theta(A - B - 2iH + Cv^2) = 0,$$

from which we obtain a result of the form

$$a(x^2 - y^2 - \alpha^2)^2 + b(2xy - \beta^2)^2 + c(2xy - \beta^2)(x^2 - y^2 - \alpha^2) = d.$$

If, further, the fixed conics are both inscribed in a rectangle, taking the axes parallel to the sides of the rectangle, we may write $E = \lambda$, $F = \mu$, so that the system becomes

$$\theta^2\lambda^2 + \mu^2 - 2\theta(A\lambda^2 + B\mu^2 + C\nu^2 + 2H\lambda\mu) = 0,$$

and the locus reduces to the equilateral hyperbola

$$Cxy + H = 0.$$

Another method of arriving at the particular conditions subject to which the general sextic locus breaks up into two circular cubics may be indicated here.

Let the general system be written

$$\theta^2(x_1\lambda + y_1\mu + \nu)^2 + (x_2\lambda + y_2\mu + \nu)^2 - \theta(A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu) = 0$$

in tangential coördinates, and suppose that one conic of the system is a circle. This evidently subjects the two given conics to a single condition. Then if the corresponding value of θ is k, and if we put $x_1\lambda + y_1\mu + \nu = E$, $x_2\lambda + y_2\mu + \nu = F$, we must have

$$kE^2 + \frac{1}{k}F^2 - (A\lambda^2 + \text{etc.}) = (l\lambda + m\mu + n\nu)^2 + p(\lambda^2 + \mu^2);$$

for the equation on the right-hand side is the general tangential equation of a circle. Hence if $l\lambda + m\mu + n\nu = G$, which represents the centre of the circle, we get

$$A\lambda^2 + \text{etc.} = kE^2 + \frac{1}{k}E^2 - G^2 - p(\lambda^2 + \mu^2).$$

Thus the system written above becomes

$$\theta^2 E^2 + F^2 - \theta\left\{kE^2 + \frac{1}{k}E^2 - G^2 - p(\lambda^2 + \mu^2)\right\} = 0.$$

Now two conics whose tangential equations differ by $\lambda^2 + \mu^2$ (the tangential equation of I, J) are confocal. Hence the conic just written has the same foci as

$$\theta^2 E^2 + F^2 - \theta\left\{kE^2 + \frac{1}{k}F^2 - G^2\right\} = 0.$$

But the envelope of the latter conic is

$$(k^2E^2 + F^2 - kG^2)^2 - 4k^2E^2F^2 = 0,$$

which breaks up into the four factors

$$kE \pm F \pm G\sqrt{k} = 0.$$

Thus the envelope consists of four points forming a quadrilateral of which the points E, F, G are the intersections of opposite sides and diagonals. Hence we have arrived at the following result: If two conics U, V are such that a circle can be described to have double contact with both of them, then the foci of a system of conics having double contact with U, V coincide with the foci of a conic passing through four fixed points. Of course it immediately follows from this that the locus in the one case is the same as that in the other. Now if ρ_1, ρ_2, ρ_3, ρ_4 are the distances of a point from the four fixed points and l, m, n, p are the areas of the triangles formed by the latter, Sylvester showed that the locus is

$$l\rho_1 + m\rho_2 + n\rho_3 + p\rho_4 = 0,$$

a curve of the sixth order, which agrees with what we have shown in this article. But, furthermore, he demonstrated that if the four points are concyclic, this locus breaks up into two circular cubics, of which the four points are foci. And this case would appear to correspond to that in which we have shown a similar breaking up to take place, viz. when two circles can be described to have double contact with the two fixed conics.

Another case of a similar kind may be added. Suppose the two fixed conics to have a focus in common, then if α, β, γ are the perpendiculars from the three foci on a line, we may write, in tangential coördinates,

$$\Sigma = \frac{\alpha\beta}{b^2} - \Omega, \qquad \Sigma' = \frac{\alpha\gamma}{b'^2} - \Omega,$$

where $\Omega\,(=1)$ is the tangential equation of I, J. We have, then, for one of the systems of enveloping conics,

$$EF = \alpha\left(\frac{\beta}{b^2} - \frac{\gamma}{b'^2}\right),$$

and

$$\theta^2\alpha^2 - 2\theta\left(\frac{\alpha\beta}{b^2} + \frac{\alpha\gamma}{b'^2} - 2\Omega\right) + \left(\frac{\beta}{b^2} - \frac{\gamma}{b'^2}\right)^2 = 0.$$

But this conic is confocal with

$$\theta^2\alpha^2 - 2\theta\alpha\left(\frac{\beta}{b^2}+\frac{\gamma}{b'^2}\right)+\left(\frac{\beta}{b^2}-\frac{\gamma}{b'^2}\right)^2=0,$$

which may be written

$$\sqrt{\theta\alpha}+\frac{1}{b}\sqrt{\beta}+\frac{1}{b'}\sqrt{\gamma}=0,$$

which represents a conic passing through α, β, γ, and touching a given tangent at the first of these points. Now this is a particular case of the system of conics passing through four fixed points; viz. it is obtained by supposing two of the points to coincide and determine a definite direction. Thus the locus is reduced to a particular case of Sylvester's result. Hence, also, it will follow that if the tangent

$$\alpha=0,\ \frac{\beta}{b^2}-\frac{\gamma}{b'^2}=0$$

is the tangent to the circumscribing circle of α, β, γ at α, then the locus breaks up into the two circular cubics of which α is the node and β, γ the foci.

In the case of spherical conics there are somewhat similar results, but they are less simple and interesting, as necessarily follows from the fact that a spherical conic has, in addition to the two real foci, two pairs of imaginary ones; that is, not counting opposite points as different foci. This is in consequence of the points I, J being replaced by the imaginary circle at infinity. Properties, however, which do not depend upon the *absolute* breaking up into factors remain the same. Thus if a system of sphero-conics be described to have double contact with a fixed sphero-conic U and a fixed circle V, a pair of their foci (not necessarily real) will lie on a sphero-conic confocal with U and passing through a pair of the intersections of the common tangent arcs of U and V.

Again suppose we consider a system of sphero-conics having double contact with two sphero-conics which both have double contact with another conic Σ. Writing the two conics

$$\Sigma=(P+Q)^2,\ \Sigma=(P-Q)^2,$$

in tangential coördinates, so that P, Q are intersections of common tangents, the system may be written

$$\frac{P^2}{\cos^2\theta}+\frac{Q^2}{\sin^2\theta}=\Sigma,$$

for the envelope of this with regard to θ breaks up into the factors $\Sigma=(P\pm Q)^2$. Now suppose, as we have already done for a particular case *in plano*, that Σ is a circle; then

$$\Sigma=R^2-\Omega,$$

where R is the centre, and Ω is the imaginary circle at infinity. Thus the system becomes

$$\frac{P^2}{\cos^2\theta}+\frac{Q^2}{\sin^2\theta}=R^2-\Omega;$$

that is, a conic confocal with

$$\frac{P^2}{\cos^2\theta}+\frac{Q^2}{\sin^2\theta}=R^2.$$

Thus the locus of the foci of the system coincides with the similar locus for a system of sphero-conics passing through four fixed points. The latter lies on a cone of the sixth degree, but when the four points are concyclic, does not break up into factors, as takes place in the plane case. This locus can easily be found as follows: let $\rho_1, \rho_2, \rho_3, \rho_4$ be the arcual distances of any arbitrary point of the sphere from four fixed points of the surface, then we can evidently determine l, m, n, p so that the relation

$$l\cos\rho_1+m\cos\rho_2+n\cos\rho_3+p\cos\rho_4=0$$

is identically true. This is, in fact, the determinant relation

$$\begin{vmatrix} \cos\rho_1, & \cos\rho_2, & \cos\rho_3, & \cos\rho_4 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix}=0.$$

Now let one focus satisfy this equation; then since the sum of the distances of a point of the curve from the foci is constant $=2\alpha$, say, the other focus must satisfy

$$\begin{aligned} &l\cos(2\alpha-\rho_1)+m\cos(2\alpha-\rho_2)+n\cos(2\alpha-\rho_3) \\ &\quad+p\cos(2\alpha-\rho_4)=0, \end{aligned}$$

or
$$\begin{aligned} &\cos 2\alpha(l\cos\rho_1+m\cos\rho_2+n\cos\rho_3+p\cos\rho_4) \\ &\quad+\sin 2\alpha(l\sin\rho_2+m\sin\rho_2+n\sin\rho_3+p\sin\rho_4)=0, \end{aligned}$$

which in consequence of the identity reduces to

$$l\sin\rho_1+m\sin\rho_2+n\sin\rho_3+p\sin\rho_4=0.$$

This, if cleared of radicals, would at first sight lead to a cone of the eighth degree, but it is easily seen to be divisible by $x^2+y^2+z^2$; for, putting

$$\sin\rho_1=\sqrt{\{x^2+y^2+z^2-(xx_1+yy_1+zz_1)^2\}}, \text{ etc.},$$

and then $x^2+y^2+z^2=0$, we get the identical relation.

If the four points are concyclic, we can easily find the

resulting relation; for, putting $\rho_1 = \rho_2 = \rho_3 = \rho_4$ for the centre of the circle in the identity, we get

$$l + m + n + p = 0;$$

but combine this as we may with

$$\Sigma l \cos \rho = 0, \quad \Sigma l \sin \rho = 0,$$

it does not seem possible to demonstrate that the locus breaks up into factors.

With respect to the actual geometrical forms of the locus in the preceding cases, it would obviously be a work of great length and detail to enumerate and classify them in order. Besides considering the base conics with regard to their being elliptic or hyperbolic, and with regard to the reality or otherwise of their intersections, it would also be of primary importance to determine whether special forms of conics of the system existed, and whether they were real or imaginary. Thus if a parabola of the system is real, the locus has in general two real asymptotes, and if there is no real parabola, the locus has no real part at infinity. Similarly, if a circle belong to the system, its centre will give rise to a node on the locus, and, moreover, one at which the tangents are at right angles, as we see by considering the imaginary foci as well as the real. Further, in all cases the two pairs of common tangents of the base conics, into which two conics of the system break up for special values of the parameter, give rise to two nodes. The tangents at the latter are evidently the bisectors of the angle between the common tangents, and, if real, are the limiting states of the axes of hyperbolas.

It is to be observed that the locus, or part of the locus, may have no direct geometrical signification; that is, points may satisfy the equation of the locus without being foci of any real conic; for instance, the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0,$$

which is entirely imaginary, envelopes the two real conics

$$\frac{a}{x} + \frac{b}{y} = 1,$$

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1,$$

and has two real foci situated at $x = 0$, $y^2 = a^2 - b^2$, where $a > b$. It seems thus that there may be a real locus for an entirely imaginary system of conics.

NOTE ON THE COMMON TANGENTS OF TWO SIMILAR CYCLOIDAL CURVES.

BY PROFESSOR F. MORLEY.

In the *Educational Times* for November, 1895, appears a problem by Professor Aiyar,* from which it appears that from among the common tangents of two similar cycloids (not merely *right* cycloids, but epi- and hypocycloids) we can so select n (where n is the class of the curves) that three of them determine the rest; and that these n common lines touch a conic. This implies that all the common tangents, — say there are μn of them, where μ is to be found, — break up into μ sets of n, and determine μ conics. The fact of the common tangents separating into sets is interesting and, I think, new; and I propose *to determine the configuration formed by the μ conics*, using a method explained in the *American Journal* (vols. 15 and 16), which avoids difficulties which we should meet if we worked solely with the well-known line-equation of the curve; namely,

$$\delta = c \cos m\omega.$$

Working with the hypocycloid, we write its equation

$$(p-q)x/c = pt^{-q} - qt^{p}, \tag{1}$$

where p, q are positive integers with no common factor, and $p > q$. Here t is a complex number of absolute value 1, say a "turn"; and x is therefore complex. The real constant c is the radius of the vertex circle; and $t = 1$ gives a vertex. The equation, in fact, maps the unit circle $|t| = 1$ into the hypocycloid. The equation conjugate to (1) is

$$(p-q)y/c = pt^{q} - qt^{-p},$$

and this and (1) are parametric equations of the curve.

The tangent at the point t of the curve is

$$xt^{q} + qt^{p} = c(1 + t^{p+q}). \tag{2}$$

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* Professor RAMASWAMI AIYAR, M.A. — A variable epi- or hypocycloid X similar to a given one touches three given straight lines continuously. Show that (1) the locus of the centre of X is a straight line; (2) X also touches $(n-3)$ other fixed lines where n is the class of the curve; (3) all these n lines are tangents to a conic; and (4) the envelope of the *vertex-circle* of X is the same conic. [In particular, if a cycloid extended both ways touches three given straight lines continuously, it touches an infinite number of other lines, all of which are tangents to a parabola; the envelope of the vertex-line of the cycloid is the same parabola.]

This is the line-equation of the curve, and is its own conjugate. To introduce homogeneous line-coördinates, we can write for any line

$$x\xi + y\eta = \zeta;$$

then if δ, ω are the distance from the origin to that line, and the inclination of that distance to the line of zero-direction,

$$\xi : \eta : \zeta = \exp - i\omega : \exp + i\omega : 2\delta, \tag{3}$$

for the equation of the line is

$$X\cos\omega + Y\sin\omega = \delta,$$

where $$X + iY = x, \quad X - iY = y;$$

$$i.e.\ x(\cos\omega - i\sin\omega) + y(\cos\omega + i\sin\omega) = 2\delta.$$

We have, then, for our curve

$$\xi : \eta : \zeta = t^q : t^p : c(1 + t^{p+q}), \tag{4}$$

from which we can find a homogeneous line-equation if we want one.

We have supposed the curve conveniently placed, but we have to handle two such curves. We observe, then, that the most general form is obtained by writing $\tau(x - \gamma)$ for x, where τ is a turn and γ is the new centre. Thus we may take two similar hypocycloids as given by the line-equations

$$\tau x t^q + y t^p/\tau = c(1 + t^{p+q}),$$

$$\tau_1(x - \gamma) t_1^q + (y - \gamma) t_1^p/\tau_1 = c_1(1 + t_1^{p+q}),$$

where the zero of direction is the line of centres from 0 to γ, τ and τ_1 are turns which bring the curves to where they are from the positions in which vertices lie on the zero of direction, c and c_1 are the vertex-radii.

To obtain common tangents, we identify these equations, and have

$$\frac{\tau t^q}{\tau_1 t_1^q} = \frac{\tau_1 t^p}{\tau t_1^p} = \frac{c(1 + t^{p+q})}{c_1(1 + t_1^{p+q}) + \gamma(\tau_1 t_1^q + t_1^p/\tau_1)} = k,$$

suppose, where k is a turn.

Hence, eliminating t_1, we have

$$kc_1(1 + t^{p+q}/k^2) + \gamma(\tau t^q + t^p/\tau) = c(1 + t^{p+q});$$

that is, $$(c - c_1/k) t^{p+q} + c - c_1 k = \gamma(\tau t^q + t^p/\tau) \tag{5}$$

where $$k^{p-q} = (\tau/\tau_1)^{p+q}. \tag{6}$$

We have, then, $p-q$ values of k, and for each value $p+q$ points of the first curve, the tangents at which touch the second curve. Thus the number of common tangents of the hypocycloids is p^2-q^2.

But for the epicycloid we have to change the sign of q throughout. We shall have, then, by the same analysis, $p+q$ values of k, but (5) is still of degree $p+q$; so that the number of common tangents of two similar epicycloids is $(p+q)^2$, the full number for curves of class $p+q$. This distinction being borne in mind, the rest of the work applies as well to the epicycloid as to the hypocycloid for which it is stated.

The equation (5), which determines a set Γ_{p+q} of $p+q$ points, contains three constants, c_1, γ, k, depending on the arbitrary second curve. Therefore three of the points, and not fewer, determine the others. If the roots are not all of the form $\exp i\theta$, this merely means that some of the $p+q$ points are imaginary points of the curve.

We now find the conic which, as Professor Aiyar states, touches the tangents at Γ_{p+q}; and we shall work with the homogeneous line-coördinates defined as in (3). We have, namely,

$$\frac{\zeta}{\tau t^q}=\frac{\eta}{t^p/\tau}=\frac{\zeta}{c(1+t^{p+q})}=\frac{\zeta-\gamma(\xi+\eta)}{c_1(k+t^{p+q}/k)},$$

or, if

$$\zeta_1=\zeta-\gamma(\xi+\eta),$$

$$\frac{\xi}{\tau t^q}=\frac{\eta}{t^p/\tau}=\frac{c_1k\zeta-c\zeta_1}{cc_1(k-1/k)t^{p+q}}=\frac{c\zeta_1-c_1\zeta/k}{cc_1(k-1/k)},$$

therefore,

$$\begin{aligned}c^2c_1^2(k-1/k)^2\xi\eta&=\gamma(c_1k\zeta-c\zeta_1)(c\zeta_1-c_1\zeta/k)\\&=cc_1\zeta\zeta_1(k+1/k)-c_1^2\zeta^2-c^2\zeta_1^2.\end{aligned}\qquad(7)$$

In terms of δ, δ_1, the distances from the centre to a tangent, this is, from (3),

$$c^2c_1^2\sin^2\alpha=c_1^2\delta^2+c^2\delta_1^2-2cc_1\delta\delta_1\cos\alpha\qquad(8)$$

where

$$k=\exp i\alpha.$$

Written in the form

$$(c\delta_1-c_1\delta\cos\alpha)^2=c_1^2\sin^2\alpha(c^2-\delta^2),\qquad(9)$$

the equation shows that the conic given by (7) or (8) has double contact with the vertex-circle $\delta=c$; and similarly it has double contact with the second vertex-circle $\delta_1=c_1$.

There are, of course, $p-q$ conics, which arise from the $p-q$ values of k given by (6). To determine how they lie, we will first find the real foci. It is a convenient general principle that when a curve is expressed (as in (8)) by means

of the distances $\delta_1, \delta_2 \cdots$ of a tangent from fixed points $a_1, a_2 \cdots$, the real foci are the roots of the equation found by writing, in the *highest terms* of the line-equation, $f - a_1$ for δ_1, $f - a_2$ for δ_2, etc. Applying this to (8), we have for the real foci the quadratic

$$c_1^2 f^2 + c^2(f-\gamma)^2 - 2cc_1 f(f-\gamma)\cos\alpha = 0,$$

where f is a stroke from the centre of the first cycloid. Hence the two foci are given by

$$\begin{aligned} c_1 f/c(f-\gamma) &= \cos\alpha \pm i\sin\alpha \\ &= k \text{ or } 1/k; \end{aligned} \tag{10}$$

i.e. they are

$$f = \frac{c\gamma}{c - c_1 k}, \quad \bar{f} = \frac{c\gamma}{c - c_1/k}. \tag{11}$$

The inverses of the foci f as to the circle (c) are

$$g = c^2/\bar{f} = \frac{c^2 - cc_1/k}{\gamma}, \tag{12}$$

and therefore form a regular $(p-q)$-*ad* (vertices of a regular polygon) in a circle whose centre is c^2/γ, and radius is cc_1/γ. Therefore the foci f themselves form a periodic $(p-q)$-*ad*, or "harmonic polygon," whose Hessian points or centres of inversion are the centres of the curves.

A point g lies both on the directrix, corresponding to f, of the conic whose focus is f, and on the chord of contact of that conic with the vertex-circle (c). For the directrix is at a distance a/e from the line of centres, where $2a$ is the focal axis, e the eccentricity; now from (8), when $\delta = \delta_1 = a$, we have

$$a^2 = \frac{c^2 c_1^2 \sin^2\alpha}{c^2 + c_1^2 - 2cc_1\cos\alpha},$$

and from (10),

$$iea = \tfrac{1}{2}(f - \bar{f}) = \frac{icc_1\gamma\sin\alpha}{c^2 + c_1^2 - 2cc_1\cos\alpha};$$

therefore,

$$a/e = cc_1\sin\alpha/\gamma,$$

and this is the ordinate of g.

And again, to locate the chord of contact, we have, from (9), the equation of its pole,

$$c_1\delta\cos\alpha = c\delta_1,$$

whence the pole is

$$x = c\gamma/(c - c_1\cos\alpha),$$

and its inverse as to (c) is

$$(c^2 - cc_1\cos\alpha)/\gamma,$$

which is the abscissa of g.

The inverse g_1 of a focus f as to the second vertex-circle (c_1) is similarly a point on a circle with centre $c_1 - c_1^2/\gamma$ and radius cc_1/γ; but observe that the angle α_1 made by that radius of this circle which ends at g_1 with the line from γ to 0 is the *opposite* of the angle α which was the corresponding angle for g_1 made by the radius from c^2/γ to g with the line from 0 to γ. For α was determined by the equation (6):

$$(\exp i\alpha)^{p-q} = (\tau/\tau_1)^{p+q};$$

α_1 is determined similarly by the equation

$$(\exp i\alpha_1)^{p-q} = (-\tau_1/-\tau)^{p+q};$$

whence for each α there is an α_1 such that

$$\alpha + \alpha_1 = 0.$$

These points g, then, appear the proper basis for viewing the system of conics, which can be constructed as follows: We start with the two vertex-circles of radii c, c_1, and with centres 0, γ. We take the inverse of each centre as to the other circle, and with these inverse points as centres, we describe equal circles of radius cc_1/γ. These are coaxial with the given circles (as will be shown), and are, being equal, equidistant from the radical axis R. In the circle whose centre is c^2/γ, we place any regular $(p-q)$-*ad* of points g; and we reflect this in R so as to get a regular $(p-q)$-*ad* of points g_1 in the other equal circle. Taking any pair of points g, g_1, let the lines $\overline{0g}$, $\overline{\gamma g_1}$ meet at f. The conic whose focus is f, whose directrix is gg_1, and whose centre is on the line of centres $\overline{0\gamma}$, is one of the system. The $p-q$ pairs of points g, g_1 give by the same construction the $p-q$ conics.

It remains to prove the elementary fact that the equal circles are coaxial with the given ones (c) and (c_1). The foci of all conics which are bitangent with two circles, when they are conjugate as to the line of centres, lie on the circle of similitude of (c) and (c_1). This was remarked by Roberts (Examples on Conics, p. 86), and, in fact, appears from (10), for from (10)

$$|f/(f-\gamma)| = |c/c_1|;$$

that is, the circles (c) and (c_1) appear of the same size from all points f. Now we know that the circle of similitude is coaxial with (c) and (c_1); that is, it passes through their real or imaginary intersections. Naturally, then, its inverse as to either circle (c) or (c_1) will pass through the same intersections. That is, the fact is evident for real intersections; but an algebraic proof would apply as much to imaginary as to real intersections.

The conic (7) depends solely on k; that is, on τ/τ_1; that is, on the relative orientation of the two curves; but τ, and therefore the actual orientation of the first curve, disappeared in finding the conic. Hence if we now let the two curves rotate about their centres with equal angular velocities, the envelope of their common tangents is the entire system of conics.

HAVERFORD, PA.

NOTES.

THE Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York on Friday afternoon, December 27, at three o'clock, the President, Dr. HILL, in the chair. There were twenty-five members present. On the recommendation of the Council, the following persons, nominated at the preceding meeting, were elected to membership: Mr. JULIUS L. NEUFELD, Central High School, Philadelphia, Pa.; Mr. WENDELL MELVILLE STRONG, Yale University, New Haven, Conn. Reports were presented by the Secretary and Treasurer. The Secretary stated that the membership of the Society was 267. The average attendance at the ordinary meetings during the year had been 15, the attendance at the last annual meeting 17, and that at the summer meeting 26. The number of members who had attended at least one meeting during the year was 61. Professor Cole, Dr. Chittenden, and Professor Ladue were appointed an auditing committee to examine the Treasurer's accounts.

The chair appointed Mr. Legras and Mr. Lambert tellers for the annual election. Upon examining the ballots cast in person and by mail, they announced that the following ticket had been elected:

President:	Dr. GEORGE W. HILL.
Vice-President:	Professor HUBERT A. NEWTON.
Secretary:	Professor FRANK N. COLE.
Treasurer:	Professor R. S. WOODWARD.
Librarian:	Professor POMEROY LADUE.

Committee of Publication:
Professor THOMAS S. FISKE.
Professor ALEXANDER ZIWET.
Professor FRANK MORLEY.

Three Members of the Council to serve until December, 1898:
Professor E. W. HYDE.
Professor W. WOOLSEY JOHNSON.
Professor B. O. PEIRCE.

After the election, Dr. HILL delivered the presidential address. It was entitled:

"Remarks on the progress of celestial mechanics since the middle of the century."

The following paper was read also:

Professor JAMES MCMAHON: "Note on the separation of the velocity potential (expressed by functions of Laplace and Bessel) into two parts, representing an outward and an inward moving wave."

THE annual meeting of the London Mathematical Society was held on November 14. The following officers were elected: President, Major P. A. Macmahon; Vice-Presidents, Professor M. J. M. Hill, Mr. M. Jenkins, Mr. A. B. Kempe; Treasurer, Dr. J. Larmor; Secretaries, Mr. R. Tucker, Mr. A. E. H. Love; Other members of the Council, Mr. H. F. Baker, Mr. G. H. Bryan, Lieut.-Col. A. J. Cunningham, Professor E. B. Elliott, Dr. J. W. L. Glaisher, Professor A. G. Greenhill, Dr. E. W. Hobson, Professor W. H. Hudson, Mr. F. S. Macaulay. According to *Nature*, Mr. Jenkins, who retires on grounds of health from the office of Secretary, has served the Society in that capacity for thirty years, he and the late Professor de Morgan having been elected joint Secretaries at the annual meeting held January 15, 1866.

THE following courses in higher mathematics will be delivered in Cambridge (Eng.) during the Lent term, 1896. In pure mathematics: Professor Forsyth, Abel's theorem and Abelian functions, Calculus of variations; Dr. Glaisher, Elliptic functions; Mr. Baker, Theory of functions (automorphic functions); Mr. Bennett, Ruled surfaces, congruences and complexes; Mr. Pendlebury, Theory of numbers. In applied mathematics: Professor Stokes, Physical optics; Professor J. J. Thomson, Electricity and magnetism; Professor Sir R. S. Ball, Planetary theory; Dr. Hobson, Higher dynamics; Mr. Larmor, Electro-dynamics and electro-optics; Mr. Herman, Hydrodynamics; Mr. Glazebrook, Sound.

AT the Anniversary Meeting of the Royal Society of London, held Saturday, November 30, 1895, the Copley medal was conferred on Professor Karl Weierstrass in recognition of his epoch-making contributions to mathematics.

IT appears from the "Compte rendu du bureau local du COMITÉ LOBATCHEFSKY, 1893–1895" (comp. BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY, vol. 2, p. 193, and vol. 3, p. 201), that the total sum collected by the Lobachevsky Committee of the University of Kazàn, up to the 13th of

May, 1895, amounts to about 24,500 frcs., of which 85% was contributed by Russia (32% by Kazàn) and 2% by the United States. The main portion of this sum has been used to establish an international mathematical prize of 500 rubles (about $270), to be awarded every three years by the Physico-Mathematical Society of Kazàn. The first award will take place November 3, 1897. No special problems will be set, the prize being given for the most valuable work on geometry, especially non-Euclidean, published in the course of the six preceding years, and sent to the Society at least one year before the date set for awarding the prize.

A smaller portion of the sum collected will be used to contribute to the erection of two busts of Lobachevsky, one in the precincts of the University, the other in a public square of the city of Kazàn. The unveiling of the latter statue will take place in September 1896. The Committee invites all scientific men interested in Lobachevsky's work to attend the unveiling; all persons accepting this invitation will be considered the guests of the city and university of Kazàn.

It is also intended to use a small part of the Lobachevsky fund to issue a new edition of his works, which will soon become necessary.

Finally, the Physico-Mathematical Society of Kazàn has decided to form a *Bibliotheca Lobatchevskiana*, and requests all workers in non-Euclidean geometry to send copies of their works or papers to this collection. A list of the volumes now on hand (about 80) is appended to the "Compte rendu."

BESIDES the prize problem (for 1897) reported in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, Vol. I (1894), p. 22, the PRINCE JABLONOWSKI SOCIETY of Leipsic offers another prize of 1000 marks, to be awarded in 1899, for the treatment of the following subject:

"Considering that the theory of Newtonian forces as developed by Poisson, Green, Gauss, Dirichlet, and others, forms one of the most important branches of mathematical physics, that, however, there are various objections to the assumption that Newton's law must hold in all cases (especially for very small and very large distances), it appears desirable that the theory of actions at a distance should be developed with greater generality, other laws of action at a distance than Newton's being taken into consideration.

"An attempt of this kind was made by Green, as early as in 1832, in his 'Mathematical investigations concerning the laws of equilibrium of fluids analogous to the electric fluid' (*Transactions of the Cambridge Philosophical Society*, 1833, reprinted in G. GREEN, *Mathematical Papers*, pp. 117–183). Forces acting according to the general law $1/r^n$, instead of

Newton's law $1/r^2$, are here taken into consideration. This important and acute investigation suggests, however, various questions that still remain unsettled or obscure; to some of these Green himself calls attention. Moreover, certain problems, for instance, the problem of electrical distribution on an ellipsoid or on a circular plate, are only briefly touched upon by Green. The Society desires, therefore, that *the problems merely indicated by Green in this paper should be actually solved, and any omissions and obscurities occurring in the paper should be supplied and cleared up.*"

The manuscripts should be in the hands of the Secretary, Professor F. Zirkel (Leipsic, Thalstrasse 33), by the 30th of November, 1898.

ERRATA. On page 3, line 25, of this volume of the BULLETIN (October No.) the formula

$$y = 2rs(p^2 + q^2) \pm 2rs^2(p^2 - q^2)$$

should be written thus:

$$y = 2rs(p^2 + q^2) \pm 2s^2(p^2 - q^2).$$

On page 9, line 9 from the bottom, for the word *five* the word *four* should be substituted.

NEW PUBLICATIONS.

HIGHER MATHEMATICS.

ABHANDLUNGEN ZUR GESCHICHTE DER MATHEMATIK. Heft VII: J. L. HEIBERG, Ptolemæus de analemmate; M. CURTZE, Ein Beitrag zur Geschichte der Algebra in Deutschland im fünfzehnten Jahrhundert; M. CURTZE, Die Handschrift Nr. 14836 der Königl. Hof- und Staatsbibliothek zu München; F. RUDIO, Eine Autobiographie von Gotthold Eisenstein, mit ergänzenden biographischen Notizen; A. HURWITZ und F. RUDIO, Briefe von G. Eisenstein an M. A. Stern; A. WASSILJEF, Nikolaj Iwanowitsch Lobatschefskij, Rede, gehalten bei der feierlichen Versammlung der Kaiserlichen Universität Kasan am 22. Oktober 1893, aus dem Russischen übersetzt von F. Engel. (Supplement zum 40sten Jahrgang der *Zeitschrift für Mathematik und Physik*.) Leipzig, Teubner, 1895. 8vo. 244 pp. 1 plate. Mk. 7.60

BALL (W. W. R.). A primer of the history of mathematics. London and New York, Macmillan, 1895. 12mo. 162 pp. Cloth. $0.65

BOLYAI (J.). La science absolue de l'espace, indépendante de la vérité ou de la fausseté de l'axiome XI d'Euclide (que l'on ne pourra jamais établir *a priori*); suivi de la quadrature géométrique du cercle dans le cas de la fausseté de l'axiome XI. Traduit de l'allemand par J. Hoüel. Nouvelle édition. Paris, Hermann, 1896. 8vo. Fr. 4.00

Bosscha (J.). Christian Huygens. Rede, gehalten am 200sten Gedächtnistage seines Lebensendes. Mit erläuternden Anmerkungen vom Verfasser. Aus dem Holländischen übersetzt von T. W. Engelmann. Leipzig, Engelmann, 1895. 8vo. 78 pp. Mk. 1.60

Brahy (E.). Exercices méthodiques de calcul intégral. Paris, Gauthier-Villars, 1895. 8vo. 8 and 302 pp. Fr. 5.00

Byerly (W. E.). Problems in differential calculus, supplementary to a treatise on differential calculus. Boston, Ginn, 1895. 12mo. 8 and 72 pp. Cloth. $0.80

Curtze (M.). See Abhandlungen zur Geschichte der Mathematik.

Darboux (G.). Sur une classe remarquable de courbes et de surfaces algébriques et sur la théorie des imaginaires. 2e tirage. Paris, Hermann, 1896. 8vo. 352 pp. Fr. 12.00

Demartres. Cours d'analyse de la Faculté des sciences de Lille. 3e partie: Équations différentielles et aux dérivées partielles. Paris, Hermann, 1895. 4to. (Lithogr.) Fr. 8.00

Dickson (L. E.). A quadratic Cremona transformation defined by a conic. (*Rendiconti del Circolo matematico di Palermo*, vol. 9, fasc. 5, pp. 256–259.) 1895. 8vo.

Elliott (E. B.). An introduction to the algebra of quantics. (Clarendon Press series.) London and New York, Macmillan, 1895. 8vo. 15s.

Engel (F.). See Abhandlungen zur Geschichte der Mathematik, and Wassiljef (A.).

Engelmann (T. W.). See Bosscha (J.).

Fleury (H.). L'analyse dite infinitésimale sans limites ni infiniment petits. 2e édition, augmentée de quatre notes fort importantes. Paris, 1896. 8vo. 80 pp. Fr. 2.00

Freycinet (C. de). Essais sur la philosophie des sciences. Analyse; mécanique. Paris, Gauthier-Villars, 1896. 8vo. 12 and 336 pp. Fr. 6.00

Heiberg (J. L.). See Abhandlungen zur Geschichte der Mathematik.

Hurwitz (A.). See Abhandlungen zur Geschichte der Mathematik.

Killing (W.). Bemerkungen über Veronese's transfinite Zahlen. [Progr.] Giessen, 1895. 4to. 30 pp.

Kœnigs (G.). Mémoire sur les lignes géodésiques. (Couronné par l'Académie des sciences.) Paris, Imprimerie nationale, 1895. 4to. 318 pp. Fr. 10.00

Lechalas (G.). Étude sur l'espace et le temps. Paris, Alcan, 1896. 18mo. 206 pp.

Picard (É.). Sur les groupes de transformations des équations différentielles linéaires. (*Comptes rendus*, vol. 119.) Paris, 1894. 4to. 6 pp.

——— Sur la théorie des surfaces et des groupes algébriques. (*Comptes rendus*, vol. 120.) Paris, 1895. 4to. 3 pp.

RUDIO (F.). See ABHANDLUNGEN ZUR GESCHICHTE DER MATHEMATIK.

SEGRE (C.). Sulla forma Hessiana. (*Rendiconti della R. Accademia dei Lincei*, vol. 4, 2d semester, 5th series, pp. 143–148.) 1895. 4to.

——— Cenni su "Die Entwickelung der Theorie der algebraischen Funktionen in älterer und neuerer Zeit, von A. Brill und M. Noether." (*Atti della R. Accademia delle scienze di Torino*, vol. 30.) Torino, Clausen, 1894. 8vo. 5 pp.

TABER (H.). On those orthogonal substitutions that can be generated by the repetition of an infinitesimal orthogonal substitution. (*Proceedings of the London Mathematical Society*, vol. 26, pp. 364–376.) 1895. 8vo.

TABER (H.). On the automorphic transformation of an alternate bilinear form. (*Mathematische Annalen*, vol. 46, pp. 561–583.) 1895. 8vo.

VOGT (H.). Leçons sur la résolution algébrique des équations. Avec une préface de J. Tannery. Paris, Nony, 1895. 8vo. 8 and 202 pp. Fr. 5.00

WASSILJEF (A.). Nikolaj Iwanowitsch Lobatschefskij. Rede, gehalten bei der feierlichen Versammlung der Kaiserlichen Universität Kasan am 22. Oktober 1893. Aus dem Russischen übersetzt von F. Engel. (*Abhandlungen zur Geschichte der Mathematik*, Heft VII, pp. 207–244.) Leipzig, Teubner, 1895. 8vo.

ELEMENTARY MATHEMATICS.

ADAMS (H.). Practical trigonometry for the use of engineers, architects, and surveyors. London, Whittaker, 1895. 8vo. 62 pp. 2s. 6d.

ARMSTRONG (H. F.). See CUSACK'S Solid geometry.

BORK (H.). Mathematische Hauptsätze für Gymnasien, methodisch zusammengestellt. 2ter Teil: Pensum des Obergymnasiums (bis zur Reifeprüfung). Leipzig, Dürr, 1895. 8vo. 235 pp. Mk. 2.40

CUSACK'S Solid geometry, in which orthographical projection is fully treated as required by the various certificates of the Science and Art Department, for which science subject I. is required. By H. F. Armstrong. London, City of London Book Dépôt, 1895. 4to. 252 pp. 3s. 6d.

DUCRUE (J.). See STAHEL.

EUCLID. See NIXON (R. C. J.) and TAYLOR (H. M.).

KIESSLING (J.). See SIMON (M.).

KLIMASZEWSKI (J.). Solution du problème de la quadrature du cercle. Paris, Reiff, 1895. 8vo. 12 pp.

KÖSTLER (H.). Leitfaden der ebenen Geometrie für höhere Lehranstalten. Heft I: Kongruenz. 4te Auflage. Halle, Nebert, 1895. 8vo. 4 and 66 pp. Boards. Mk. 1.25

LOMBOLT (A.). Matematiske Opgaver. A: Eksamensopgaver i Regning og Matematik fra "Almindelig Forberedelseseksamen." Juni 1883–Juni 1893. 5te, utvidede udgave. Viborg, 1895. 8vo. 60 pp. Mk. 1.00

MENUGE (C.). Petites tables de logarithmes à cinq décimales pour les nombres et pour les sinus et tangentes, précédées des règles du calcul par logarithmes et de notions pratiques de trigonométrie, avec les applications à la mesure des distances et à l'arpentage. 5e édition. Paris, Delagrave, 1896. 12mo. 48 pp.

NIXON (R. C. J.). Euclid revised; containing the essentials of the elements of plane geometry as given by Euclid in his first six books. With numerous additional propositions and exercises. 3d edition. (Clarendon Press series.) London and New York, Macmillan, 1895. 8vo. 448 pp. 6s.

——— Euclid revised. Part II: containing the essentials of Euclid's fifth and sixth books. With additional propositions, a compendium of modern geometry, and numerous exercises. 3d edition. (Clarendon Press series.) London and New York, Macmillan, 1895. 8vo. 218 pp. 3s. 6d.

RULAND (N.). Praktische Anleitung zum gründlichen Unterricht in der Buchstabenrechnung. Ausführliche Auflösung der in Dr. E. Heis' Sammlung von Beispielen u. s. w. enthaltenen Aufgaben. Teil 1: Die allgemeine Arithmetik und Algebra. Zum Selbstunterricht bestimmt. 6te Auflage. Bonn, Cohen, 1895. 8vo. 12 and 460 pp. Mk. 6.00

SIMON (M.) und KIESSLING (J.). Didaktik und Methodik des Rechnen-, Mathematik-, und Physik-Unterrichts. (Aus A. Baumeister's Handbuch der Erziehungs- und Unterrichtslehre für höhere Schulen.) München, Beck, 1895. 8vo. 128 and 73 pp. Mk. 4.50

STAHEL'S Sammlung von Prüfungsaufgaben. II: Die Absolutorialaufgaben in Bayern. Heft I: Aufgaben aus der Mathematik und Naturwissenschaft an den humanistischen Gymnasien, Realgymnasien und Realschulen Bayerns. Als Uebungsstoff für den Repetitionsunterricht zusammengestellt von J. Ducrue. 6te Auflage, ergänzt bis 1895. Würzburg, Stahel, 1895. 12mo. 2 and 116 pp. Boards. Mk. 1.00

TAYLOR (H. M.). Euclid's Elements of geometry. Books XI and XII. Edited for the syndics of the Pitt Press. (Pitt Press mathematical series.) Cambridge University Press (Macmillan), 1895. 12mo. 168 pp. 1s. 6d.

WENTWORTH (G. A.). Plane and spherical trigonometry. Revised edition. Boston, Ginn, 1895. 8vo. 6 and 192 and 24 pp. $0.85

APPLIED MATHEMATICS.

BAUER (P.). Das elektromagnetische Kraftlinienfeld eines kreisförmigen Oscillators. [Diss.] München, 1895. 8vo. 56 pp.

BIESE (A. C.). Theorie der Fernrohre mit continuierlich variabler Vergrösserung. Berlin, Fussinger, 1895. 8vo. 30 pp. Mk. 2.00

BIGOURDAN (G.). Inventaire général des manuscrits de la bibliothèque de l'observatoire de Paris. Paris, 1895, 4to. Fr. 3.00

BREUER (A.). Mathematische Vorschule der Astronomie in Bezug auf die scheinbare Bewegung des Fixsternhimmels. Wien, 1895. 8vo. 24 pp. 3 plates. Mk. 1.80

CHRISTIE (A. S.). The latitude-variation tide. (*Philosophical Society of Washington, Bulletin*, vol. 13, pp. 103–122.) Washington, 1895. 8vo.

DOYÈRE (C.). Notions élémentaires de mécanique du navire. Partie II: Examen d'application. Paris, 1895. 8vo. Cloth. Fr. 4.00

DUHEM (P.). Le potentiel thermodynamique et ses applications à la mécanique chimique et à l'étude des phénomènes électriques. 2e édition. Paris, Hermann, 1895. 8vo. 300 pp. Fr. 10.00

FAYE (H.). Sur l'origine du monde. Théorie cosmogoniques des anciens et des modernes. 3e édition, revue et corrigée. Paris, Gauthier-Villars, 1896. 8vo. 12 and 314 pp. Fr. 6.00

FRANK (M.). Das thermo-elektrostatische Potential. Untersuchungen über die Art der Beziehung zwischen Wärme und Elektrizität. München, Finsterlin, 1895. 8vo. 50 pp. Mk. 1.50

FRITSCHE (H.). Ueber den Zusammenhang zwischen der erdmagnetischen Horizontalintensität und der Inclination. St. Petersburg, Fritsche, 1895. 8vo. 14 pp. 29 plates. Mk. 3.00

GALILEI (G.). Le opere. Edizione nazionale sotto gli auspizi di Sua Maestà il Re d' Italia. Vol. V: Proposte per la determinazione della longitudine. Firenze, Barbèra, 1895. 4to. 430 pp.

GRUNMACH (L.). Lehrbuch der magnetischen und electrischen Maasseinheiten, Messmethoden und Messapparate. Stuttgart, Enke, 1895. 8vo. 16 and 632 pp. Mk. 16.00

HAUSDORFF (F.). Ueber die Absorption des Lichtes in der Atmosphäre. [Habilitationsschrift.] Leipzig, 1895. 8vo. 83 pp.

HEDERICH. Anleitung zur Ausführung graphischer Konstruktionen im Maschinenbau. Heft I: Zahnräder und Achsen. Jena, Costenoble, 1895. 8vo. 4 and 43 pp. 9 plates. Mk. 5.00

HINRICHS (J.). Beiträge zu der Bewegung eines materiellen Punktes auf der Oberfläche eines gleichseitigen, ein- oder zweischaligen Rotations-Hyperboloids unter der Einwirkung einer Centralkraft. [Diss.] Marburg, 1895. 8vo. 56 pp. 3 plates.

HOPPE (O.). Das Maschinenwesen. Elementares Lehrbuch zur Einführung in die Maschinenwissenschaften, die Kinematik und die Elastizitäts- und Festigkeitslehre. Leipzig, Felix, 1895. 8vo. 16 and 152 pp. Mk. 5.40

KLUG (J.). Inversion und elektrischer Strom. [Progr.] Würzburg 1895. 8vo. 44 pp.

LE CORGUILLÉ (T.). Étude rationnelle des marées. Rennes, Simon, 1895. 8vo. 78 pp. Plates.

LIMB (C.). Mesure directe des forces électro-motrices en unités absolues électro-magnétiques. [Thèse.] Paris, Gauthier-Villars, 1895. 4to. 8 and 120 pp.

LOMBARDI (L.). Fenomeni di polarizzazione in un campo elettrostatico uniforme. Misura di differenze di potenziale e di costanti dielettriche. [Diss.] Zürich, 1895. 4to. 68 pp. 2 plates.

MANEUVRIER (G.). Nouvelle méthode de détermination du rapport C/c pour l'air et d'autres gaz. Paris, Gauthier-Villars, 1895. 4to. 8 and 228 pp.

OLIVIER (J. VON). Was ist Raum, Zeit, Bewegung, Masse? Was ist die Erscheinungswelt? München, Finsterlin, 1895. 8vo. 60 pp. Mk. 1.20

PERREAU (F.). Étude expérimentale de la dispersion et de la réfraction des gaz. [Thèse.] Paris, Gauthier-Villars, 1895. 4to. 66 pp.

POINCARÉ (H.). Sur les équations de la physique mathématique. Paris, Hermann, 1894. 8vo. 100 pp. Fr. 5.00

PRESTON (E. D.). Determinations of latitude, gravity, and the magnetic elements at stations in the Hawaiian Islands, including a result for the mean density of the earth, 1891, 1892. (Appendix 12 of the *Report for 1893 of the U. S. Coast and Geodetic Survey*, pp. 510–639.) Washington, 1894.

REULEAUX (F.). Der Konstrukteur. Ein Handbuch zum Gebrauch beim Maschinen-Entwerfen. 4te Auflage, 3ter Abdruck. Braunschweig, Vieweg, 1895. 8vo. 72 and 1198 pp. Mk. 25.00

SCHWERER. Étude sur la loi des tempêtes. Paris, 1895. 8vo. Fr. 1.50

SEIDEMANN (K.). Ein Beitrag zur Theorie der Bewegung eines materiellen Punktes auf einer Rotationsfläche. Ein Doppelproblem, welches die elliptischen Funktionen in allen möglichen Formen zur Anwendung bringt. [Diss.] Leipzig, 1895. 8vo. 36 pp.

THIERY (A.). Ueber geometrisch-optische Täuschungen. Leipzig, 1895. 8vo. 66 pp. Mk. 1.50

THOMPSON (S. P.). Die dynamoelektrischen Maschinen. Ein Handbuch für Studierende der Elektrotechnik. Deutsche Uebersetzung von C. Grawinkel; nach dem Tode des Uebersetzers besorgt von K. Strecker und F. Vesper. (In 12 Heften.) Halle, Knapp, 1895. 8vo. Heft I: pp. 1–64. Mk. 2.00

WEISBACH (J.). Tafel der vielfachen Sinus und Cosinus, sowie der vielfachen Sinus versus von kleinen Winkeln, nebst Tafel der einfachen Tangenten, zum Gebrauche für praktische Geometer und Mechaniker überhaupt und für Markscheider besonders. 5te Ausgabe. Berlin, Weidmann, 1895. 8vo. 28 pp. Mk. 1.00